AD-A186 427 ON THE CHARACTERIZATION OF CERTAIN POINT PROCESSES(U) 1/1
NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS
T HSING AUG 87 TR-199 AFOSR-TR-87-1064 F49620-85-C-0144

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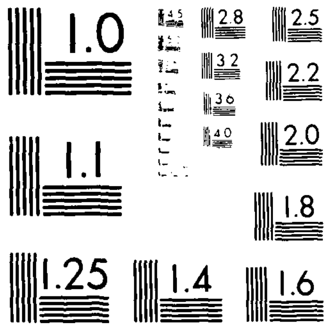

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AD-A186 427

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) Technical Report No. 199	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 87-1064
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina / 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code) 321-A Phillips Hall 039-A, Statistics Dept. Chapel Hill, NC 27514	7b. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR / 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER ~~AFOSR No~~. F49620 85C 0144
8c. ADDRESS (City, State, and ZIP Code) AFOSR/NM Bldg 410 Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. 61102F / PROJECT NO. 2304 / TASK NO. A5 / WORK UNIT ACCESSION NO.

DTIC SELECTE SEP 3 0 1987 S E

11. TITLE (Include Security Classification)
On the characterization of certain point processes

12. PERSONAL AUTHOR(S)
Dr. Tailen Hsing

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
preprint	FROM 9/86 TO 8/87	August 1987	22

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Key Words and Phrases: Extreme values, infinite divisibility, point processes, weak convergence.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj. Brian Woodruff	22b. TELEPHONE (Include Area Code) (202) 767-5026 / 22c. OFFICE SYMBOL NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete. — SECURITY CLASSIFICATION OF THIS PAGE

19 Abstract (cont'd.)

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CENTER FOR STOCHASTIC PROCESSES

AFOSR-TR- 87-1064

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

ON THE CHARACTERIZATION OF CERTAIN POINT PROCESSES

by

Tailen Hsing

Technical Report No. ~~196~~ [illegible]

August 1987

ON THE CHARACTERIZATION OF CERTAIN POINT PROCESSES

by Tailen Hsing

Texas A&M University

and

The University of North Carolina at Chapel Hill

Summary. This paper consists of two parts. First, a characterization is obtained for a class of infinitely divisible point processes on $\mathbb{R} \times \mathbb{R}_+' = (-\infty,\infty)\times(0,\infty)$. Second, the result is applied to identify the weak limit of the point process N_n with points $(j/n, u_n^{-1}(\xi_j))$, $j=0,\pm1,\pm2,\ldots$ where $\{\xi_j\}$ is a stationary sequence satisfying a certain mixed condition Δ, and $\{u_n\}$ is a sequence of non-increasing functions on $(0,\infty)$ such that $\lim_{n\to\infty} P\{\max_{1\le j\le n} \xi_j \le u_n(\tau)\} = e^{-\tau}$, $\tau>0$. This application extends a result of Mori [14], which assumes that $\{\xi_j\}$ is α-mixing, and that the distribution of $\max_{1\le j\le j} \xi_j$ can be linearly normalized to converge to a maximum stable distribution.

AMS 1980 Subject classification: Primary 60F05, 60G55, Secondary 60G10.

Key words and phrases: Extreme values, infinite divisiblity, point processes, weak convergence.

Research sponsored by the Air Force Office of Scientific Research Grant No. AFOSR F49620 85 C 0144.

1. Introduction.

It is well known that point process methods can be applied effectively to study certain types of problems in statistical extreme value theory. Consider a strictly stationary sequence of random variables $\{\xi_j\}$ indexed by the set of integers I=Z. One can define a number of interesting point processes in one dimension by recording the positions where "extreme values" occur. For example, an extremal process (cf. Dwass [4] and Lamperti [9]) typically is one that records the indices (properly normalized) at which record values of $\xi_1, \xi_2, \ldots$ occur, and an exceedance point process considered by Leadbetter [11] consists of the set of points $\{j/n\colon \xi_j > u_n,\ j=1,\ldots,n\}$ where $\{u_n\}$ is a suitable sequence of constants. For this type of processes, Poisson or compound Poisson convergence results (cf. [7], ~~[11]~~) can often be derived under suitable mixing conditions.

It is also useful to consider certain point processes in two dimensions in this context. A number of authors studied the point process η_n consisting of the points $(j/n, a_n^{-1}(\xi_j - b_n))$, $j \in I$, where $a_n > 0$, b_n are constants such that $P\{\max_{1 \le i \le n} \xi_i \le a_n x + b_n\}$ converges weakly to some nondegenerate distribution function $G(x)$. In this connection, Poisson convergence of η_n was first established by Pickands [17] for i.i.d. $\{\xi_j\}$ (cf. also Resnick [18]). Adler [1] studied the conditions under which the point process η_n performs as one generated by an i.i.d. sequence when n becomes large, Mori [14] identified all possible limit laws of η_n assuming that $\{\xi_j\}$ is α-mixing (also known as strong-mixing), and Weissman [21] considered the convergence of η_n when the ξ_j are independent but not identically distributed. Some authors also considered this type of point processes using nonlinear normalizations; for example, both Hsing [5] and Leadbetter et al. [12] considered the point process with points

$(j/n, u_n^{-1}(\xi_j))$, $j \in I$, where u_n is such that $\lim_{n\to\infty} nP[\xi_1 > u_n(\tau)] = \tau$ for each $\tau > 0$. Weak convergence results involving these point processes are often conveniently termed "complete convergence" theorems (cf. [12]) since they usually provide all the asymptotic distributions of the extreme order statistics with respect to the relevant normalization procedures. Rootzén [19] derived complete convergence results for a special class of processes. Davis and Resnick [3] demonstrated how information can be extracted from a complete convergence theorem and be used for the purpose of statistical inference in general.

We are especially interested in the characterization technique developed by Mori [14]. It was shown there that if $\{\xi_j\}$ is α-mixing, then the weak limit of the point process η_n mentioned previously has a specific form which is determined by a Poisson process and the "local" dependence structure of $\{\xi_j\}$. (Unfortunately the significance of [14] is masked by the presence of several crucial errors of a typographical nature.) The main purpose of the present paper is to show that this type of characterization extends to a substantially larger class of point processes (not necessarily related to extreme value theory) under reasonably simple and general conditions. In particular, the main theorem (Theorem 1) of Mori [14] will follow under conditions generalized in two directions:

(a) a much weaker mixing condition,

(b) using normalizations that are not required to be linear.

However, we attempt to present the salient features of the general theory in a transparent way so that its potential for other application will be evident to the reader.

We proceed according to the following outline. In section 2 we review the concepts of point process theory and some weak convergence results which are required. Section 3 gives the main characterization method (Theorem 3.6 and

Corollary 3.7) and section 4 applies the results to give the improved version (Theorem 4.5) of Mori [14], Theorem 1.

Finally our debt to the work of Mori [14] will be obvious and is acknowledged here rather than by repeated reference.

2. Some Useful Concepts from the Theory of Point Processes.

For clarity, we devote this section to a brief review of certain point process concepts which are particularly relevant to our theory. The reader is referred to Kallenberg [8] and Matthes et al. [13] for details.

Let S be a locally compact second countable and Hausdorff topological space. Write $\mathcal{S}$ for the Borel σ-field, and $\mathcal{B}$ the collection of all bounded (relatively compact) sets in $\mathcal{S}$. Also denote by $\mathcal{F}$ the class of nonnegative $\mathcal{S}$ measurable funtions.

A point process η on $(S,\mathcal{S})$ is a random element in M (or, for clarity, M(S)), the space of locally finite integer-valued measures on $(S,\mathcal{S})$ equipped with the vague topology and Borel σ-field $\mathcal{M}$. For each $f\in\mathcal{F}$, write ηf for the random variable $\int_S f d\eta$. If $f=1_B$ is the indicator function of a set B in $\mathcal{S}$, write $\eta(B)$ or ηB instead of $\eta 1_B$ for convenience. The distribution of η is uniquely determined by its Laplace transform $L_\eta(f)=E\exp(-\eta f)$, $f\in\mathcal{F}$.

A point process η is infinitely divisible if for each n=1,2,..., there exist some independent and identically distributed point processes $\eta_n,\ldots,\eta_n$ such that $\eta \overset{d}{=} \eta_1+\ldots+\eta_n$. The following result is important.

Theorem 2.1 (cf. [8], Theorem 6.1). The relation

$$-\log L_\eta(f) = \int_{M\setminus\{o\}}[1-\exp(-\mu f)]\lambda(d\mu) \tag{2.1}$$

defines a unique correspondence between the distributions of all infinitely divisible point processes η on $(S,\mathcal{S})$ and the class of measures λ on $M\setminus\{o\}$ (o being the null measure) satisfying

$$\int_{M\setminus\{o\}}[1-\exp(-\mu B)]\lambda(d\mu) < \infty,\ B\in\mathcal{B}.$$

λ is customarily referred to as the canonical measure of η, and (2.1) the canonical representation of L_η.

Using (2.1), many interesting properties of infinitely divisible point processes can be conveniently derived. In particular, the following is of special interest to us.

<u>Lemma 2.2</u> Let η be an infinitely divisible point process on $(S,\mathcal{S})$ with canonical measure λ. Then

(i) $P\{\eta(E) = 0\} = \exp(-\lambda\{\mu\in M\setminus\{o\}: \mu(E)>0\})$, $E\in\mathcal{S}$ (cf. [13], Lemma 2.2.5);

(ii) for any pairwise disjoint sets $E_1,\ldots,E_k$ in $\mathcal{S}$ with $P\{\sum_1^k\eta(E_i) < \infty\} > 0$, $\eta(E_1),\ldots,\eta(E_k)$ are mutually independent if and only if $\lambda\{\mu\in M\setminus\{o\}:\mu(E_i) > 0,\ \mu(E_j) > 0\} = 0$ for all i,j satisfying $1 \leq i < j \leq k$ (cf. [8], Lemma 7.3 and [13], Proposition 2.2.12).

A sequence of point processes $\{\eta_n\}$ is said to converge in distribution to some point process η if $P \circ \eta_n^{-1}$ converges weakly to $P \circ \eta^{-1}$ in the usual sense (cf. [2]) where, here and hereafter, "$\circ$" denotes the composition operation of functions. The following criterion is convenient.

<u>Theorem 2.3</u> (cf. [8], Theorem 4.2 and Lemma 4.4). Let $\eta,\eta_1,\eta_2,\ldots$ be point processes on $(S,\mathcal{S})$. Then η_n converges in distribution to η if and only if $L_{\eta_n}(f) \to L_\eta(f)$, as $n \to \infty$, for all bounded measurable functions f in $\mathcal{F}$ with bounded supports and such that $\eta\{s\in S: f \text{ is discontinuous at } s\} = 0$ with probability one.

3. A Characterization Result for Point Processes on $\mathbb{R} \times \mathbb{R}_+'$.

We now restrict our attention to point processes η on $\mathbb{R} \times \mathbb{R}_+' = (-\infty,\infty) \times (0,\infty)$. $\mathbb{R} \times \mathbb{R}_+'$ is assumed to be equipped with the usual topology and σ-field. Write $(M,\mathcal{M})$ for the space of integer-valued locally finite measures on $\mathbb{R} \times \mathbb{R}_+'$ as described in Section 2.

First, define two types of transformation which play important roles in this paper. For each $\tau\in\mathbb{R}$, $\sigma\in\mathbb{R}_+'$, let g_τ and h_σ be mappings on $\mathbb{R} \times \mathbb{R}_+'$ to $\mathbb{R} \times \mathbb{R}_+'$ defined by

$$g_\tau(x,y) = (x+\tau,y),\ h_\sigma(x,y) = (x/\sigma,\sigma y),\ (x,y) \in \mathbb{R} \times \mathbb{R}_+'.$$

Also, instead of creating different notation, g and h denote the corresponding set mappings.

For convenience, a point process η is said to satisfy (A1), (A2), (A3), or (A4) if η satisfies the respective restrictions described as follows.

(A1) $\eta \circ g_\tau \overset{d}{=} \eta$ for each $\tau \in \mathbb{R}$.

(A2) $\eta \circ h_\sigma \overset{d}{=} \eta$ for each $\sigma \in \mathbb{R}_+'$.

(A3) $P\{\eta([0,1)\times(0,\epsilon)) > 0\} \to 0$ as $\epsilon \to 0$, (or equivalently, $P\{\eta([0,1)\times(0,\epsilon)) < \infty\} = 1,\ \epsilon>0)$.

(A4) For any choice $I_1,\ldots,I_k$ of disjoint intervals of the form $[a,b)$ in $\mathbb{R}$, and any choice $J_1,\ldots,J_m$ of intervals of the form $[c,d)$ in $\mathbb{R}_+'$, the m-dimensional random vectors $(\eta(I_i\times J_1),\ldots,(I_i\times J_m))$, $i=1,2,\ldots,k$, are mutually independent, where k,m are arbitrary positive integers

The conditions (A1) - (A4) are quite stringent. As we shall soon see, a point process which satisfies all four of these conditions must be a member of a very restricted class. We commence with a simple, yet quite useful lemma.

<u>Lemma 3.1</u> If a point process η satisfies (A1) and (A2), then $\eta(\{x\}\times\mathbb{R}_+') = 0$ a.s. and $\eta(\mathbb{R} \times \{y\}) = 0$ a.s. for each $x \in \mathbb{R}$, $y \in \mathbb{R}_+'$.

Proof. Let $b > a > 0$ be arbitrary. If $P\{\eta(\{x\}\times[a,b)) > 0\} > 0$ for some x in $\mathbb{R}$, then $P\{\eta(\{x\}\times[a,b)) > 0\} > 0$ for all x in $\mathbb{R}$ by (A1). This contradicts the requirement that the set $\{x \in \mathbb{R}: P\{\eta(\{x\}\times[a,b)) > 0\} > 0\}$ must be countable (cf. [13], 1.1.5). Hence for each $x \in \mathbb{R}$,

$$P\{\eta(\{x\}\times\mathbb{R}_+^{\cdot}) > 0\} = \lim_{\substack{a\to 0\\ b\to\infty}} P\{\eta(\{x\}\times[a,b)) > 0\} = 0.$$

The other half can be shown similarly. □

Theorem 3.2 A point process η satisfying (A1) and (A4) is infinitely divisible.

Proof. It suffices to show that $\sum_{m=1}^{k} \eta(E_m)$ is infinitely divisible for each choice of positive integer k and sets $E_1,\ldots,E_k$ of the form $[a,b)\times[c,d)$ in $\mathbb{R} \times \mathbb{R}_+^{\cdot}$ (cf. [8], Lemma 6.3). Note that $\sum_{m=1}^{k} \eta(E_m)$ can be written as $\sum_{i=1}^{s} \sum_{j=1}^{t_i} \eta(E_{ij})$ with $E_{ij} = [a_i,b_i) \times [c_{ij},d_{ij})$, $i=1,\ldots,s$, $j=1,\ldots,t_i$, where the $[a_i,b_i)$ are disjoint intervals. Further for each i,j, and each positive integer n, $\eta(E_{ij})$ can be written as $\sum_{\ell=1}^{n} \eta(E_{ij}^{\ell})$ with $E_{ij}^{\ell} = [a_i + \frac{(b_i-a_i)(\ell-1)}{n}, a_i + \frac{(b_i-a_i)\ell}{n}) \times [c_{ij},d_{ij})$. Hence for each positive integer n, $\sum_{m=1}^{k} \eta(E_m) = \sum_{\ell=1}^{n} \sum_{i=1}^{s} \sum_{j=1}^{t_i} \eta(E_{ij}^{\ell})$ where, by (A1) and (A4), $\sum_{i=1}^{s} \sum_{j=1}^{t_i} \eta(E_{ij}^{\ell})$, $\ell = 1,2,\ldots,n$, are independent and identically distributed random variables. The result follows. □

Lemma 3.3 Suppose η satisfies the conditions (A1) - (A4). Then for each $y > 0$, $P\{\eta([0,1)\times(0,y)) = 0\} > 0$, and hence, by Lemma 2.2, $\lambda\{\phi \in M\setminus\{o\}: \phi([0,1)\times(0,y)) > 0\} < \infty$ where λ is the canonical measure of η.

Proof. Let $y > 0$ be arbitrary but fixed. By (A3), there exists a positive integer k such that $P\{\eta([0,1)\times(0,y/k)) = 0\} > 0$. Note that the random variables $\eta([i-1,i)\times(0,y/k))$, $i = 1,\ldots,k$, are independent by (A3), (A4), and are identically distributed by (A1). These together with (A2) imply that

$$\begin{aligned} &P\{\eta([0,1)\times(0,y)) = 0\} = P\{\eta([0,k)\times(0,y/k)) = 0\} \\ &= P\{\eta([i-1,i)\times(0,y/k)) = 0,\ i = 1,\ldots,k\} \\ &= P^k(\eta([0,1)\times(0,y/k)) = 0\} > 0. \quad \square \end{aligned}$$

Write M_1 for the collection of integer-valued locally finite measures ψ on $[1,\infty)$ such that $\psi\{1\} \geq 1$, and $\mathcal{M}_1$ its usual σ-field. Denote by ϵ_z, $z \in [1,\infty)$, and $\delta_{(x,y)}$, $(x,y) \in \mathbb{R} \times \mathbb{R}_+'$, the Dirac measures on $[1,\infty)$ and $\mathbb{R} \times \mathbb{R}_+'$, respectively. Write $(\mathbb{R} \times \mathbb{R}_+') \times M_1$ for the product space of $\mathbb{R} \times \mathbb{R}_+'$ and M_1, and introduce a mapping Ω on $(\mathbb{R} \times \mathbb{R}_+') \times M_1$ into $M\setminus\{o\}$ by

(3.1) $$\Omega:\ ((x,y),\psi) \to \Sigma a_i \delta_{(x,yz_i)}$$

where $(x,y) \in \mathbb{R} \times \mathbb{R}_+'$ and $\psi = \Sigma a_i \epsilon_{z_i} \in M_1$. Ω is obviously one-to-one and measurable. Further, since $(\mathbb{R} \times \mathbb{R}_+') \times M_1$ and $M\setminus\{o\}$ are both Polish (cf. [13], 15.7.7), Kuratowski's Theorem (cf. [16]) implies that Ω maps measurable sets to measurable sets. Write Λ for the range of Ω.

<u>Lemma 3.4</u> Suppose η is a point process satisfying the conditions (A1), (A3), and (A4). Then η is infinitely divisible, and the canonical measure λ concentrates on Λ, i.e. $\lambda(\Lambda^c) = 0$.

<u>Proof</u>. Since λ is a measure on $M\setminus\{o\}$, it is understood that all set operations are performed on this space. It is easily seen that $\Lambda \doteq A \cap B$ where A is the event $\{\phi \in M\setminus\{o\}: \phi(\{x\}\times\mathbb{R}_+') = 0$ for all but one x in $\mathbb{R}\}$, and B the event $\{\phi \in M\setminus\{o\}: \phi(\mathbb{R}\times(0,\epsilon)) = 0$ for some $\epsilon > 0\}$. Since $\Lambda^c = A^c \cup (A\cap B^c)$, it suffices to show that $\lambda(A^c) = \lambda(A\cap B^c) = 0$. Write $A_{mn} = \{\phi \in M\setminus\{o\}: \phi([\frac{k}{2^n}, \frac{k+1}{2^n})\times(0,m)) = 0$ for all but possibly one k in $I\}$ where I is the set of integers. Observe that A_{mn} is monotonically non-increasing in m for each fixed n, $\bigcap_{m=1}^{\infty} A_{mn}$ is also

monotonically non-increasing in n, and $A = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{mn}$. Thus

$$\lambda(A^c) = \lambda\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{mn}^c\right) = \lim_{n\to\infty} \lim_{m\to\infty} \lambda(A_{mn}^c)$$

(3.2)

$$\leq \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i\neq j} \lambda\{\phi \in M\backslash\{o\}: \phi([\tfrac{i}{2^n}, \tfrac{i+1}{2^n})\times(0,m)) > 0, \quad \phi([\tfrac{j}{2^n}, \tfrac{j+1}{2^n})\times(0,m)) > 0\}.$$

The conditions (A3), (A4) imply that $\eta([\frac{i}{2^n}, \frac{i+1}{2^n})\times(0,m))$ and $\eta([\frac{j}{2^n}, \frac{j+1}{2^n})\times(0,m))$ are independent if $i\neq j$, and therefore the right hand side of (3.2) equals zero by Lemma 2.2. Similarly, since $A \cap B^c \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \{\phi \in M\backslash\{o\}: \phi([-m,m)\times(0,\frac{1}{n})) > 0\}$, it follows from (A1), (A3), (A4), Lemma 2.2, and Lemma 3.3 that

$$\lambda(A \cap B^c) \leq \lim_{m\to\infty} \lim_{n\to\infty} \lambda\{\phi \in M\backslash\{o\}: \phi([-m,m)\times(0,\tfrac{1}{n})) > 0\}$$

$$= - \lim_{m\to\infty} \lim_{n\to\infty} \log P\{\eta([-m,m)\times(0,\tfrac{1}{n})) = 0\}$$

$$= - \lim_{m\to\infty} \lim_{n\to\infty} \log P^{2m}\{\eta([0,1)\times(0,\tfrac{1}{n})) = 0\} = 0,$$

concluding the theorem. □

<u>Lemma 3.5</u> A measure υ on $\mathbb{R} \times \mathbb{R}_+^{\cdot}$ satisfying $\upsilon \circ g_\tau = \upsilon \circ h_\sigma = \upsilon$ for each $(\tau,\sigma) \in \mathbb{R} \times \mathbb{R}_+^{\cdot}$ is a constant multiple of Lebesgue measure.

<u>Proof</u>. It suffices to show that $\upsilon([a,b) \times [c,d)) = (b-a)(d-c)\upsilon([0,1)\times(0,1))$, $[a,b)\times[c,d) \subset \mathbb{R} \times \mathbb{R}_+^{\cdot}$. Using $\upsilon \circ g_\tau = \upsilon$, $\tau \in \mathbb{R}$, it is easily seen that

$$\upsilon([0,1)\times(0,1)) = \sum_{k=1}^{m} \upsilon([\tfrac{k-1}{m}, \tfrac{k}{m}) \times (0,1))$$

$$= m\upsilon([0,1/m) \times (0,1))$$

for each $m\geq 1$. Thus

$$\upsilon([0,\tfrac{n}{m})\times(0,1)) = \tfrac{n}{m}\upsilon([0,1)\times(0,1))$$

for each m,n ≥ 1. By this and the assumption we have for any $[a,b)\times[c,d) \subset \mathbb{R} \times \mathbb{R}_+'$ that

$$\begin{aligned}\nu([a,b)\times[c,d)) &= \nu([0,(b-a)d)\times(0,1)) - \nu([0,(b-a)c)\times(0,1)) \\ &= \lim_{\substack{n/m\to(b-a)d \\ n,m\geq 1}} \nu([0,\tfrac{n}{m})\times(0,1)) - \lim_{\substack{n/m\to(b-a)c \\ n,m\geq 1}} \nu([0,\tfrac{n}{m})\times(0,1)) \\ &= ((b-a)d-(b-a)c)\ \nu([0,1)\times(0,1)) \\ &= (b-a)(d-c)\nu([0,1)\times(0,1)). \quad \square\end{aligned}$$

We now combine our somewhat disconnected discussion to give the following characterization.

Theorem 3.6 A point process η on $\mathbb{R} \times \mathbb{R}_+'$ satisfies (A1) - (A4) if and only if it is infinitely divisible, and there exists a probability measure Q on $(M_1, \mathcal{M}_1)$ such that the canonical measure λ on η satisfies $\lambda = \theta\ (m\times Q)\circ\Omega^{-1}$ where Ω is defined by (3.1), $\theta = -\log P\{\eta([0,1)\times(0,1)) = 0\} < \infty$, and $m \times Q$ is the product measure of Lebesgue measure m on $\mathbb{R} \times \mathbb{R}_+'$ and Q.

Proof. We first prove the "only if" part. Suppose η satisfies (A1) - (A4). It follows from Lemma 3.3 that θ is finite. If $\theta = 0$, then the result is trivially true. Assume henceforth that $\theta>0$. For each set E in $\mathcal{M}_1$, define a set function ν_E on $\mathcal{S}$, the Borel σ-field of $\mathbb{R} \times \mathbb{R}_+'$, by

$$\nu_E(B) = \lambda \circ \Omega(B\times E),\ B \in \mathcal{S}.$$

ν_E is a measure since λ is a measure and Ω is one-to-one and bi-measurable. For each $\tau \in \mathbb{R}$, write G_τ for the transformation

$$G_\tau: \phi \to \phi \circ g_{-\tau},\ M\backslash\{o\} \to M\backslash\{o\}.$$

It is evident that $\lambda \circ G_\tau = \lambda$ since the former is the canonical measure of the point process $\eta \circ g_\tau$, and $\eta \circ g_\tau \overset{d}{=} \eta$ by (A1). Also it is straightforward to verify that $\Omega(g_\tau(x,y),\psi) = G_\tau \circ \Omega((x,y),\psi)$, $\tau \in \mathbb{R}$, $(x,y) \in \mathbb{R} \times \mathbb{R}_+'$, $\psi \in M_1$. Hence for each $B \in \mathcal{S}$,

$$\nu_E \circ g_\tau(B) = \lambda \circ \Omega(g_\tau(B)\times E) = \lambda \circ G_\tau \circ \Omega(B\times E)$$
$$= \lambda \circ \Omega(B\times E) = \nu_E(B).$$

This shows that $\nu_E \circ g_\tau = \nu_E$ for each τ in $\mathbb{R}$. One could similarly show that $\nu_E \circ h_\sigma = \nu_E$ for each $\sigma \in \mathbb{R}_+'$ using (A2). With these, it follows from Lemma 3.5 that ν_E is a constant multiple of Lebesgue measure m; i.e.

(3.3) $$\lambda \circ \Omega(B\times E) = \nu_E(B) = \theta m(B)Q(E), \ B\in\mathcal{S},$$

for some constant Q(E) in $[0,\infty]$. It is clear that $Q(\emptyset) = 0$, and that if $\{E_i\}$ is a countable collection of disjoint sets in $\mathcal{M}_1$, then

$$Q(\cup E_i) = \lambda\circ\Omega(B\times\cup E_i)/(\theta m(B))$$
$$= \Sigma\ \lambda \circ \Omega(B\times E_i)/(\theta m(B))$$
$$= \Sigma Q(E_i)$$

where $B\in\mathcal{S}$ is any set for which $0<m(B) < \infty$. Thus Q is a measure. The fact that Ω maps the set $([0,1)\times(0,1))\times M_1$ to $\{\phi \in \Lambda\colon \phi([0,1)\times(0,1)) > 0\}$, and that $\lambda(\Lambda^c) = 0$ imply that

$$Q(M_1) = \lambda\circ\Omega(([0,1)\times(0,1))\times M_1)/(\theta m([0,1)\times(0,1)))$$
$$= \lambda\{\phi \in \Lambda:\phi([0,1)\times(0,1)) > 0\}/\theta$$
$$= -\log P\{\eta([0,1)\times(0,1))=0\}/\theta=1,$$

showing that Q is a probability measure. The conclusion of the "only if" part follows since (3.3) holds for each E in $\mathcal{M}_1$ and B in $\mathcal{S}$.

Having shown the "only if" part, the proof for the "if" part should be straightforward and hence we only provide a sketch. Suppose η is infinitely divisible and has the structure described in the theorem. Then (A1) and (A2) hold by virtue of the identities

$$L_{\eta\circ g_\tau}(f) = L_\eta(f\circ g_{-\tau}) = L_\eta(f),$$
$$L_{\eta\circ h_\sigma}(f) = L_\eta(f\circ h_{1/\sigma}) = L_\eta(f)$$

which follow readily from (2.1). Lemma 2.2(i) implies that

$$P\{\eta([0,1)\times(0,\epsilon))>0\} = 1-e^{-\theta\epsilon},\ \epsilon>0,$$

which, in turn, implies (A3), while (A4) follows easily from Lemma 2.2(ii). □

The following corollary states the relationship between the Poisson process and the class of point processes satisfying (A1) - (A4).

Corollary 3.7 A point process η on $\mathbb{R} \times \mathbb{R}'_+$ is infinitely divisible and has the canonical measure $\lambda=\theta(m\times Q)\circ\Omega^{-1}$ if and only if η admits the representation $\sum_{i=1}^{\infty}\sum_{j=1}^{K_i}\delta_{(S_i,T_iY_{ij})}$, where the (S_i,T_i) are the points of a homogeneous Poisson process ζ with mean θ, and, for each i, Y_{ij}, $1\leq j\leq K_i$, are the points of a point process γ on $[1,\infty)$ distributed according to Q, and $\zeta,\gamma_1,\gamma_2,\dots$ are mutually independent.

Proof. Suppose f is a nonnegative measurable function on $\mathbb{R} \times \mathbb{R}'_+$ with a bounded support E, and let ω be the point process $\sum_{i=1}^{\infty}\sum_{j=1}^{K_i}\delta_{(S_i,T_iY_{ij})}$. Conditional on $\omega(E) = k$, where k is any nonnegative integer, the points of ζ in E are independently and uniformly distributed over E. Thus

$$\begin{aligned}
L_\omega(f) &= \mathcal{E}\exp(-\int_{\mathbb{R}\times\mathbb{R}'_+} f\,d\omega) \\
&= \sum_{k=0}^{\infty} P\{\omega(E)=k\}\mathcal{E}\{\exp(-\sum_i\sum_j f(S_i,T_iY_{ij})\,|\,\omega(E)=k\} \\
&= \sum_{k=0}^{\infty} e^{-\theta m(E)}\frac{(\theta m(E))^k}{k!}\Big(\int_{M_1}\int_E e^{-\int_{0-}^{\infty} f(x,yz)\psi(dz)}\frac{m(dxdy)}{m(E)}\,Q(d\psi)\Big) \\
&= \exp\{-\theta\int_{M_1}\int_E(1-e^{-\int_{0-}^{\infty} f(x,yz)\psi(dz)})m(dxdy)Q(d\psi)\}.
\end{aligned}$$

The set E in the last expression may now be replaced by $\mathbb{R} \times \mathbb{R}'_+$ (without affecting the Laplace transform), and the expression, after a change-of-variable (via the transformation Ω), equals

$$\exp\{-\int_{M\setminus\{0\}}\{1-\exp(-\phi f)\lambda(d\phi)\},$$

which is just $L_\eta(f)$. This completes the proof. □

It is obvious from Corollary 3.7 that if each τ_i (in the representation of η) is degenerate and has only one point which is 1, η is then just a homogeneous Poisson process on $\mathbb{R} \times \mathbb{R}_+'$.

4. Point Process Associated with Extreme Observations.

Consider a strictly stationary sequence $\{\xi_j\}$ indexed by the set of integers I = Z. For each $n \geq 1$, let $M_n^{(1)} \geq M_n^{(2)} \geq \dots \geq M_n^{(n)}$ be the order statistics of $\xi_1, \dots, \xi_n$, and write, for convenience, M_n for $M_n^{(1)}$.

Throughout this section we assume the existence of a sequence $\{u_n\}_{n\geq 1}$ of functions on $\mathbb{R}_+' = (0,\infty)$ with the following properties:

(B1) For each n, u_n is nonincreasing, left continuous, and such that
$\lim_{\substack{\tau_1 \to 0 \\ \tau_2 \to \infty}} P\{u_n(\tau_2)<\xi_1<u_n(\tau_1)\}=1$.

(B2) For each $\tau > 0$, $\lim_{n\to\infty} P\{M_n \leq u_n(\tau)\} = e^{-\tau}$.

Define $u_n^{-1}(\xi) = \sup\{\tau>0: \xi \leq u_n(\tau)\}$. It is easily seen that $u_n^{-1}(\xi)<\tau$ if and only if $\xi>u_n(\tau)$. The point process of interest in this section is N_n which is a point process on $\mathbb{R} \times \mathbb{R}_+'$ with points $(j/n, u_n^{-1}(\xi_j))$, $j \in I$. Many random quantities connected with the extremes of $\{\xi_j\}$ can be studied through N_n since

$$(4.1) \qquad N_n((0,x]\times(0,\tau)) \leq k-1 \quad \text{if and only if}$$
$$M_{[nx]}^{(k)} \leq u_n(\tau), \qquad \tau, x>0,\ 1 \leq k \leq [nx].$$

We shall show in the following that the distributional limit of N_n satisfies the conditions (A1) - (A4) stated in section 3 provided that $\{\xi_j\}$ satisfies a certain mixing condition Δ which we now introduce. Let k and n be positive integers. For each choice of $\tau_1, \dots, \tau_k > 0$, and $1 \leq \ell \leq n-1$, write

$$\alpha(n,\ell;\tau_1,\dots,\tau_k) = \max\{|P(A\cap B) - P(A)P(B)|: A\in\mathcal{F}_{1,s},$$
$$B\in\mathcal{F}_{s+\ell,n}\ ,\ 1 \leq s \leq n-\ell\}$$

where F_{ij} is the σ-field generated by the events $(\xi_s \leq u_n(\tau_m))$, $i\leq s\leq j$, $1\leq m\leq k$.

The condition Δ is said to hold for $\{\xi_j\}$ and the sequence $\{u_n\}$ if for each choice of k, and $\tau_1,\ldots,\tau_k$, $\alpha(n, [\lambda n]; \tau_1,\ldots,\tau_k) \to 0$ as $n \to \infty$ for each $\lambda \in (0,1)$, where [x] denotes the integer part of x.

The condition Δ is obviously weaker than the α-mixing condition, and in practice Δ can be verified more easily than α-mixing. On the other hand, the condition Δ is potentially stronger than some distributional type mixing conditions (cf. [10], [15]) that are useful in the context of proving extremal types theorems. We use the condition Δ in this paper since it appears to be most convenient for our purpose. The way in which Δ can be modified or further weakened should become evident.

Lemma 4.1 Assume that the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$. Then for each $0 < \sigma \leq 1$ and $\tau > 0$,

$$\lim_{n\to\infty} P\{M_{[\sigma n]} \leq u_n(\tau)\} = e^{-\sigma\tau} \tag{4.2}$$

where, here and hereafter, [y] denotes the integer part of y. It can be derived from this that for $\tau>0$ and $\sigma_1>\sigma_2>0$, $u_{[n/\sigma_1]}(\tau) > u_n(\sigma_2\tau)$ for all large n.

Proof. (4.2) follows readily from some well-known results (cf. [10], [15]). For $\sigma_1 > \sigma_2 \geq 1$,

$$\lim_{n\to\infty} P\{M_{[n/\sigma_1]} \leq u_n(\sigma_1\tau)\} = e^{-\tau}, \text{ and}$$

$$\lim_{n\to\infty} P\{M_{[n/\sigma_1]} \leq u_{[n/\sigma_2]}(\tau)\} = \lim_{n\to\infty} P\{M_{[\frac{\sigma_2}{\sigma_1}[\frac{n}{\sigma_2}]]} \leq u_{[\frac{n}{\sigma_2}]}(\tau)\} \tag{4.3}$$

$$= \lim_{n\to\infty} P\{M_{[\frac{\sigma_2}{\sigma_1}n]} \leq u_n(\tau)\} = e^{-\sigma_2\tau/\sigma_1} > e^{-\tau},$$

where the first and second equality of the second equation follow, respectively, from the facts

$$\lim_{n\to\infty} P\{\max(\xi_j:\ 1\le j\le [n/\sigma_1] - [\frac{\sigma_2}{\sigma_1}[\frac{n}{\sigma_2}]]) > u_{[\frac{n}{\sigma_2}]}(\tau)\} = 0$$

and $\{[n/\sigma_2]:\ n\ge 1\} = \{n\ge 1\}$. (4.3) implies that $u_{[n/\sigma_2]}(\tau) > u_n(\sigma_1\tau)$ for large n. This conclusion holds similarly for other choices of σ_1 and σ_2 such that $\sigma_1>\sigma_2>0$. □

Lemma 4.2 Assume that the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$. Let k,m be positive integers, s_{ij}, $1\le i\le k$, $1\le j\le m$, be nonnegative integers, and x_i,τ_j, $1\le i\le k$, $1\le j\le m$, be nonnegative reals. If either

$$P\{N_n([0,\sigma x_i)\times(0,\tau_j))\le s_{ij},\ 1\le i\le k,\ 1\le j\le m\}$$

or

$$P\{N_n([0,x_i)\times(0,\sigma\tau_j))\le s_{ij},\ 1\le i\le k,\ 1\le j\le m\}$$

converges for each σ in some interval $(\sigma_\ell,\ \sigma_u)$, where $\sigma_\ell>1$, then both probabilities converge and have the same limit for each $\sigma\in(\sigma_\ell,\ \sigma_u)$.

Proof. We shall only prove the lemma for the case k=m=1, since the general situation is similarly proved. Also, for clarity of presentation, the arguments in this proof are phrased in terms of the order statistics (cf. (4.1)). In other words, we shall show that the convergence of either $P\{M_n^{(s)}\le u_n(\sigma\tau)\}$ or $P\{M_{[\sigma n]}^{(s)}\le u_n(\tau)\}$ for each $\sigma\in(\sigma_\ell,\sigma_u)$ implies the convergence of the other to the same limit for each $\sigma\in(\sigma_\ell,\sigma_u)$, where s is an arbitrary positive integer. First assume that $P\{M_n^{(s)}\le u_n(\sigma\tau)\}$ converges for each σ in (σ_ℓ,σ_u). For σ and σ' with $\sigma_\ell<\sigma<\sigma'<\sigma_u$,

$$\begin{aligned}\limsup_{n\to\infty} P\{M_{[\sigma' n]}^{(s)}\le u_n(\tau)\} &= \limsup_{n\to\infty} P\{M_{[\sigma'[n/\sigma']]}^{(s)}\le u_{[n/\sigma']}(\tau)\}\\ &= \limsup_{n\to\infty} P\{M_n^{(s)}\le u_{[n/\sigma']}(\tau)\}\le \lim_{n\to\infty} P\{M_n^{(s)}\le u_n(\sigma\tau)\}.\end{aligned}\tag{4.4}$$

Here the first equality follows from the identity $\{n:\ n\ge 1\} = \{[n/\sigma']:\ n\ge 1\}$, the second equality holds since $0\le n-[\sigma'[n/\sigma']]\le\sigma'$ and $P\{M_{[\sigma']} >$

$u_{[n/\sigma']}(\tau)\}\to 0$, and the inequality follows from *Lemma* 4.1. Similarly, for σ and σ'' with $\sigma_\ell < \sigma'' < \sigma < \sigma_u$,

$$\text{(4.5)} \qquad \liminf_{n\to\infty} P\{M^{(s)}_{[\sigma''n]}(k) \le u_n(\tau)\} \ge \lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma\tau)\}.$$

By (4.4) and (4.5), for σ and σ_i, $1 \le i \le 4$, with $\sigma_\ell < \sigma_1 < \sigma_2 < \sigma < \sigma_3 < \sigma_4 < \sigma_u$,

$$\limsup_{n\to\infty} P\{M^{(s)}_{[\sigma_4 n]} \le u_n(\tau)\} \le \lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma_3\tau)\} \le \lim_{n\to\infty} P\{M^{(s)}_n \ge u_n(\sigma\tau)\}$$
$$\le \lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma_2\tau)\} \le \liminf_{n\to\infty} P\{M^{(s)}_{[\sigma_1 n]} \le u_n(\tau)\}.$$

But

$$\liminf_{n\to\infty} P\{M^{(s)}_{[\sigma_1 n]} \le u_n(\tau)\} - \limsup_{n\to\infty} P\{M^{(s)}_{[\sigma_4 n]} \le u_n(\tau)\}$$
$$\le \limsup_{n\to\infty} (P\{M^{(s)}_{[\sigma_1 n]} \le u_n(\tau)\} - P\{M^{(s)}_{[\sigma_4 n]} \le u_n(\tau)\})$$
$$\le \lim_{n\to\infty} P\{M_{[(\sigma_4-\sigma_1)n]} > u_n(\tau)\} = 1 - e^{-(\sigma_4-\sigma_1)\tau}$$

which tends to zero if $\sigma_4 - \sigma_1 \to 0$. This shows that $\lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\cdot\tau)\}$ is *continuous at* σ. *Since for* σ,σ_1, and σ_2 with $\sigma_\ell \le \sigma_1 < \sigma < \sigma_2 < \sigma_u$,

$$\lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma_1\tau)\} \le \liminf_{n\to\infty} P\{M^{(s)}_{[\sigma n]} \le u_n(\tau)\} \le \limsup_{n\to\infty} P\{M^{(s)}_{[\sigma n]} \le u_n(\tau)\}$$
$$\le \lim_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma_2\tau)\}$$

by (4.4) and (4.5), it is easily seen that $P\{M^{(s)}_{[\sigma n]} \le u_n(\tau)\}$ converges and has the same limit as does $P\{M^{(s)}_n \le u_n(\sigma\tau)\}$.

Suppose now $P\{M^{(s)}_{[\sigma n]} \le u_n(\tau)\}$ converges for each σ in (σ_ℓ,σ_u). Using arguments similar to the ones in getting (4.4) and (4.5), it can be seen that for σ,σ_1, and σ_2 with $\sigma_\ell < \sigma_1 < \sigma < \sigma_2 < \sigma_u$,

$$\lim_{n\to\infty} P\{M^{(s)}_{[\sigma_2 n]} \le u_n(\tau)\} \le \liminf_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma\tau)\} \le \limsup_{n\to\infty} P\{M^{(s)}_n \le u_n(\sigma\tau)\}$$
$$\le \lim_{n\to\infty} P\{M^{(s)}_{[\sigma_1 n]} \le u_n(\tau)\}.$$

As before, the difference between $\lim_{n\to\infty} P\{M^{(s)}_{[\sigma_1 n]} \le u_n(\tau)\}$ and $\lim_{n\to\infty} P\{M^{(s)}_{[\sigma_2 n]} \le u_n(\tau)\}$ tends to zero as σ_1 and σ_2 tend to σ. This concludes the proof. □

Lemma 4.3 Suppose the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$, and that N_n converges in distribution to some N. Then N satisfies (A1) and (A2), and $P\{N([0,1)\times(0,\epsilon)) > 0\} = 1-e^{-\epsilon}$, $\epsilon>0$, which implies that N satisfies (A4).

Proof. That N satisfies (A1) follows readily from the stationarity of $\{\xi_j\}$. By Lemma 3.1, $N(\{x\}\times\mathbb{R}_+') = 0$ a.s. for each $x\in\mathbb{R}$. Similarly, by Theorem 1.1.5 of [13], there exists a countable set C such tht $N([0,1)\times\{\tau\}) = 0$ a.s. for each $\tau \in D:=\mathbb{R}_+'\backslash C$. Thus $N(\mathbb{R}\times\{\tau\}) = 0$ a.s., $\tau\in D$, by (A1). For $\tau < \epsilon$ in D, $[0,1)\times[\tau,\epsilon)$ is bounded and has N-a.s. zero boundary. Thus Theorem 2.3 implies that $N_n([0,1)\times[\tau,\epsilon)) \xrightarrow{d} N([0,1)\times[\tau,\epsilon))$. Since $\lim_{\epsilon\to0}\lim_{n\to\infty} P\{N_n([0,1)\times(0,\epsilon)) > 0\} = 0$, it follows from an application of [2], theroem 4.2 that

$$P\{N([0,1)\times(0,\epsilon)) > 0\} = \lim_{n\to\infty} P\{N_n([0,1)\times(0,\epsilon)) > 0\}$$
$$= \lim_{n\to\infty} P\{M_n > u_n(\epsilon)\} = 1-e^{-\epsilon}$$

for each ϵ in D, and hence, by continuity, for each $\epsilon > 0$. This shows (A3). It is clear from (A1) and (A3) that $N([0,x)\times(0,\tau)) < \infty$ a.s. for all $x,\tau>0$. Thus by (A1), Lemma 4.2, and [8], Theorem 3.1, that (A2) holds for N follows from the convergence in distribution of the random vector $(N_n([0,x_i)\times(0,\tau_j))$, $1\le i\le k$, $1\le j\le m)$ to $(N([0,x_i)\times(0,\tau_j))$, $1\le i\le k$, $1\le j\le m)$ for each choice of $x_i>0$, $\tau_j\in D$, and positive integers k and m. The convergence is easily shown using arguments similar to the ones above and is left for the reader. □

Lemma 4.4 Suppose the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$, and that N_n converges in distribution to some N. Then N satisfies (A4).

Proof. We shall prove the claim for $k = 2$. The proof for unrestricted k is similar, but more complicated notationally. Let $I_i = [a_i,b_i)$, $i=1,2$, be disjoint intervals in $\mathbb{R}$, and $J_j = [c_j,d_j)$, $j=1,2,\dots,m$, be intervals in $\mathbb{R}_+'$. It suffices to show that

$$P\{N(I_i \times J_j) = s_{ij},\ i=1,2,\ j=1,\ldots,m\} = \prod_{i=1}^{2} P\{N(I_i \times J_j) = s_{ij},\ j=1,\ldots,m\}$$

for each choice of non-negative integers s_{ij}, $i=1,2$, $j=1,\ldots,m$. For this purpose, it is important to note that, by Lemma 4.4, both I_1 and I_2 can be assumed to be in $(0,1]$ without any loss of generality. Denote by I_2' the interval $[a_2+\epsilon,\ b_2)$ where ϵ is any non-negative number less than (b_2-a_2). It follows from the triangle inequality that for each n,

$$\left|P\{N(I_i \times J_j) = s_{ij},\ i=1,2, j=1,\ldots,m\} - \prod_{i=1}^{2} P\{N(I_i \times J_j) = s_{ij}, j=1,\ldots,m\}\right| \leq \sum_{i=1}^{5} g_i(n)$$

where

$$g_1(n) = |P\{N(I_i \times J_j)=s_{ij}, i=1,2, j=1,\ldots,m\} - P\{N_n(I_j \times J_j)=s_{ij}, i=1,2, j=1,\ldots,m\}|,$$

$$g_2(n) = |P\{N_n(I_i \times J_j) = s_{ij}, i=1,2, j=1,\ldots,m\}$$
$$- P\{N_n(I_1 \times J_j)\ s_{1j}, N_n(I_2' \times J_j) = s_{2j}, j = 1,\ldots,m\}|,$$

$$g_3(n) = |P\{N_n(I_1 \times J_j) = s_{1j}, N_n(I_2' \times J_j) = s_{2j},\ j=1,\ldots,m\}$$
$$- P\{N_n(I_1 \times J_j) = s_{ij}, j = 1,\ldots,m\} P\{N_n(I_2' \times J_j) = s_{2j}, j = 1,\ldots,m\}|,$$

$$g_4(n) = P\{N_n(I_1 \times J_j) = s_{1j}, j = 1,\ldots,m\} \cdot |P\{N_n(I_2' \times J_j) = s_{2j}, j = 1,\ldots,m\}$$
$$- P\{N_n(I_2 \times J_j) = s_{2j}, j = 1,\ldots,m\}|,$$

$$g_5(n) = |P\{N_n(I_1 \times J_j) = s_{1j}, j = 1,\ldots,m\} P\{N_n(I_2 \times J_j) = s_{2j}, j=1,\ldots,m\}$$
$$- P\{N(I_1 \times J_j) = s_{1j}, j = 1,\ldots,m\} P\{N_n(I_2 \times J_j) = s_{2j}, j = 1,\ldots,m\}|.$$

Since $N_n \xrightarrow{d} N$, and using the fact by Lemma 4.3 that N satisfies (A1) and (A2), Theorem 2.3 and Lemma 3.1 imply that $g_1(n)$ and $g_2(n)$ both tend to zero as n tends to ∞. Write $d = \max_{1 \leq j \leq m} (d_j)$, and note, by Boole's inequality, that both $g_2(n)$ and $g_4(n)$ are bounded by $P\{N_n([a_2,\ a_2+\epsilon) \times (0,d)) > 0\}$, or by $P\{M_{[n\epsilon]} > u_n(d)\}$, which tends to $1-e^{-\epsilon d}$ by Lemma 4.1. Finally since the condition A hold for $\{\xi_j\}$ and $\{u_n\}$, $g_3(n)$ is bounded by $\alpha(n,[n\epsilon]+1;\ c_1,\ldots,c_k,d_1,\ldots,d_k)$, showing that $g_3(n)$ tends to zero as n tends to infinity. Summarizing the above, we get

$$\left| P\{N(I_i \times J_j) = s_{ij}, i=1,2, j=1,\ldots,m\} - \prod_{i=1}^{2} P\{N(I_i \times J_j) = s_{ij}, j=1,\ldots,m\} \right|$$

$$\le \limsup_{n\to\infty} \sum_{i=1}^{5} g_i(n) \le 2(1-e^{-\epsilon d}).$$

Letting ϵ tend to zero, the result follows. □

The main result of this paper now follows from Corollary 3.7, Lemma 4.3, and Lemma 4.4.

Theorem 4.5 Let $\{\xi_j\}$ be a strictly stationary sequence of random variables, and $\{u_n\}$ a sequence of functions on $\mathbb{R}_+'$ for which (B1) and (B2) hold. Suppose the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$, and that the point process N_n $=: \sum_{j\in I} \delta_{(j/n, u_n^{-1}(\xi_j))}$ converges in distribution to some point process N, where N_n and N are point processes on $\mathbb{R} \times \mathbb{R}_+'$. Then N has the representation $\sum_{i=1}^{\infty} \sum_{j=1}^{K_i} \delta_{(S_i, T_i Y_{ij})}$, where (S_i, T_i), $i\ge 1$, are the points of a mean one homogeneous Poisson process on $\mathbb{R} \times \mathbb{R}_+'$, Y_{ij}, $1\le j\le K_i$, are the points of a point process γ_i on $[1,\infty)$ with 1 as an atom, $\gamma_1, \gamma_2, \ldots$ are identically distributed, and $\eta, \gamma, \gamma_2, \ldots$ are mutually independent.

It is plausible to view the points Y_{ij}, $1\le j\le K_i$, of γ_i in the representation of N as describing the magnitudes (normalized by u_n^{-1}) of the members in a cluster of extreme observations of $\{\xi_j\}$, relative to the largest observation in the cluster. For the important special case where $\{\xi_j\}$ is i.i.d., extreme observations do not cluster, and thus each γ_i has only a point which is 1 (cf. [12], Theorem 5.7.2), leaving the Poisson process the only possible limit for N_n. See also Davis and Resnick [3], and Rootzén [19,20] for further justifications of this viewpoint.

The following corollary shows how Theorem 1 of [14] can be derived from Theorem 4.5.

Corollary 4.6 Suppose $\{\xi_j\}$ is α-mixing and there are constants $a_n > 0$ and b_n such that $P\{M_n \le a_n x + b_n\} \xrightarrow{w} \exp(-e^{-x})$, $x \in \mathbb{R}$. Define $\tilde{N}_n$ to be the point process on $\mathbb{R} \times \mathbb{R}$ with points $(j/n, (\xi_j - b_n)/a_n)$, $j \in I$. If $\tilde{N}_n$ converges in distribution to some $\tilde{N}$, where the weak convergence takes place in $M(\mathbb{R} \times \mathbb{R})$ (cf. Section 2), $\tilde{N}$ has points $(S_i, -\log(T_i Y_{ij}))$, $i \ge 1$, $1 \le j \le K_i$, where the (S_i, T_i) and Y_{ij} are as described in Theorem 4.5

Proof. Let $u_n(\tau) = -a_n \log\tau + b_n$, $\tau > 0$, $n \ge 1$. $\{u_n\}$ obviously satisfies (B1) and (B2) for $\{\xi_j\}$. If $\tilde{N}_n$ converges in distribution, in the space $M(\mathbb{R} \times \mathbb{R})$, to some $\tilde{N}$, then by the continuous mapping theorem $N_n := \sum_j \delta_{(j/n, u_n^{-1}(\xi_j))} =$

$\sum_j \delta_{(j/n, \exp(-(\xi_j - b_n)/a_n))}$ converges in distribution to some N, as random elements in $M(\mathbb{R} \times \mathbb{R}_+^{\cdot})$. Since α-mixing is stronger than the condition Δ, Theorem 4.5 implies that N has the representation $\sum_i \sum_j \delta_{(S_i, T_i Y_{ij})}$, which, again by the continuous mapping theorem, concludes the corollary. □

To complete this characterization, Mori [14] showed that any point process γ on $[1,\infty)$ having atoms at 1 can be a "cluster process" in the representation of $\tilde{N}$ (cf. [14], Theorem 2). Thus, in view of the proof of Corollary 4.6, the characterization of N_n in Theorem 4.5 is also complete.

Finally, it is interesting to interpret the above point process convergence in terms of extreme order statistics.

Theorem 4.7 Assume that the condition Δ holds for $\{\xi_j\}$ and $\{u_n\}$. N_n converges in distribution if and only if $P\{M_n^{(k_i)} \le u_n(\tau_i),\ 1 \le i \le m\}$ converges for each choice of $\tau_i > 0$, $k_i \ge 1$, $1 \le i \le m$, $m \ge 1$, and $\lim_{k\to\infty} \lim_{n\to\infty} P\{M_n^{(k)} \le u_n(\tau)\} = 1$ for each $\tau > 0$, where $M_n^{(k)}$ is the k^{th} maximum of $\xi_1, \ldots, \xi_n$.

Proof. Suppose first N_n converges to N. By the definition of u_n and Theorem 2.3 (cf. Lemma 4.3),

$$\lim_{n\to\infty} P\{M_n^{(k_i)} \le u_n(\tau_i),\ 1 \le i \le m\}$$
$$= \lim_{n\to\infty} P\{N_n([0,1)\times(0,\tau_i)) \le k_i-1,\ 1 \le i \le m\}$$
$$= P\{N([0,1)\times(0,\tau_i)) \le k_i-1,\ 1 \le i \le m\}.$$

Also it is clear that $N([0,1)\times(0,\tau)) < \infty$ a.s. and thus the only if part follows. Next suppose the converse is true. The assumption $1 = \lim_{k\to\infty}\lim_{n\to\infty} P\{M_n^{(k)} \le u_n(\tau)\} = \lim_{k\to\infty}\lim_{n\to\infty} P\{N_n([0,1)\times(0,\tau)) \le k\}$, $\tau > 0$, implies that the family $\{N_n: n \ge 1\}$ is tight (cf. [8], Lemma 4.5), and hence for every infinite subsequence I' of the set of positive integers, there exists a further subsequence I'' along which N_n converges in distribution to some N'. It suffices to show that the distribution of N' is independent of the choice of I' and I''. N', as a limit of N_n, has the representation obtained in Theorem 4.5, and therefore its distribution is determined by the set of probabilities

$$P\{N'([0,1)\times(0,\tau_i)) \le k_i-1,\ 1 \le i \le m\}$$
$$= \lim_{n\to\infty} P\{M_n^{(k_i)} \le u_n(\tau_i),\ 1\le i\le m\},\ \tau_i>0,\ k_i\ge 1,\ 1 \le i \le m,\ m\ge 1,$$

which are clearly independent of I' and I''. This proves that N_n converges in distribution. □

Acknowledgement.

I wish to express my gratitude to Ross Leadbetter for the infinite help he has given me. Also I wish to thank the referee for his careful reading of the paper and constructive comments and criticisms.

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